Multivariable analysis 2021-2022
Mock Exam

Below you will find several questions based on the exams from previous years (plus some extra exercises). The first 4 questions give an estimation of the size of the exam.

QUESTIONS

1. 10+10+10 = 30 pts

   i) Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Show that $f\colon \mathbb{R}^n \to \mathbb{R}$ given by $f = \|v\|^2$ is differentiable at every point $p \in \mathbb{R}^n$.

   ii) Compute the first and second differential at a point $p = (a, b)$ of the function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (xy^2 + e^{x+y}, 2y)$.

   iii) Compute the differential at the origin of

   $$G(x, y) = g^1(x, y)\colon \mathbb{R}^2 \to \mathbb{R},$$

   where $g^t(x, y)$ is the flow the linear ODE

   $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

2. 10+10 = 20 pts The set of solutions to the system of two equations given below is called $S \subset \mathbb{R}$.

   $$-xyz + 3z^3 - w^4 - 1 = 0$$
   $$2xz - yw + x^3 - 2 = 0$$

   i) Find a basis for the tangent space $T_{s_0}S$ to the solution $s_0 = (1, 1, 1, 1)$.

   ii) Use the implicit function theorem to show that close to the solution $x = y = z = w = 1$ the points of S can be written as C^1 functions of two out of the four variables.

3. 15 pts Let $\omega = dx \wedge dy \wedge dz \wedge dw$ be the standard volume form on $\mathbb{R}^4$. Compute the two-form η defined by

   $$\eta_{v_1,v_2}(u_1, u_2) = \omega(v_1, v_2, u_1, u_2),$$

   where v_1 and v_2 are vector fields on $\mathbb{R}^4$ given by $v_1 = x\frac{\partial}{\partial y}$ and $v_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$.

4. $\boxed{5+8+10+5+7 = 35 \text{ pts}}$ Define $f\colon \mathbb{R}^3 \to \mathbb{R}^4$ by $f(x, y, z) = (2x + 3y, xz, yz, xy)$ and ω, η by $\omega(a, b, c, e) = a\mathrm{d}a \wedge \mathrm{d}e + c\mathrm{d}b \wedge \mathrm{d}c$ and $\eta(a, b, c, e) = \mathrm{d}a \wedge \mathrm{d}c$.

i) Show that $f^\star(\omega \wedge \eta) = 0$.

ii) Express $f^\star\omega$ in terms of $\mathrm{d}x, \mathrm{d}y$ and $\mathrm{d}z$.

iii) Compute the integrals $\int_{\sigma_1} f^\star\omega$ and $\int_{\sigma_2} \omega$, where $\sigma_1 = (\gamma, [0, 1]^2)$ and $\sigma_2 = (f \circ \gamma, [0, 1]^2)$ with γ defined by $\gamma(s, t) = (s, t, st)$.

iv) Do you see a relation between the two answers from the previous part?

v) Apply Stokes's theorem to write $\int_{\sigma_2} \eta$ as an integral over a 1-chain.

5. (bonus) Let N be a regular C^∞-smooth k-dimensional surface in $\mathbb{R}^n$ with boundary.

i) Prove that the boundary ∂N is a regular C^∞-smooth $(k-1)$-dimensional surface in $\mathbb{R}^n$;

ii) Prove that ∂N is orientable when N is orientable (meaning that it admits a nowhere vanishing top form);

iii) Prove that a closed ball $\overline{B_r(x_0)}$ in $\mathbb{R}^n$ is a compact regular C^∞-smooth $k = n$-dimensional surface in $\mathbb{R}^n$ with boundary.

iv) Prove that a closed ball $\overline{B_r(x_0)}$ in $\mathbb{R}^n$ can be viewed as an n-cell.